Multidimensional Disjunctive Inequalities for Linear Chance-Constrained Problems with Finite Support

AIROYoung Workshop

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Université Libre de Bruxelles, Belgium

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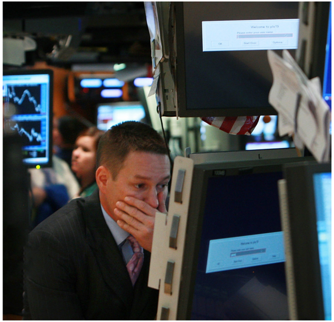
How much attention do extreme events deserve?



Chance Constrained Stochastic Programs (CCSPs)

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\text{s.t.} \quad & x \in X(\xi)
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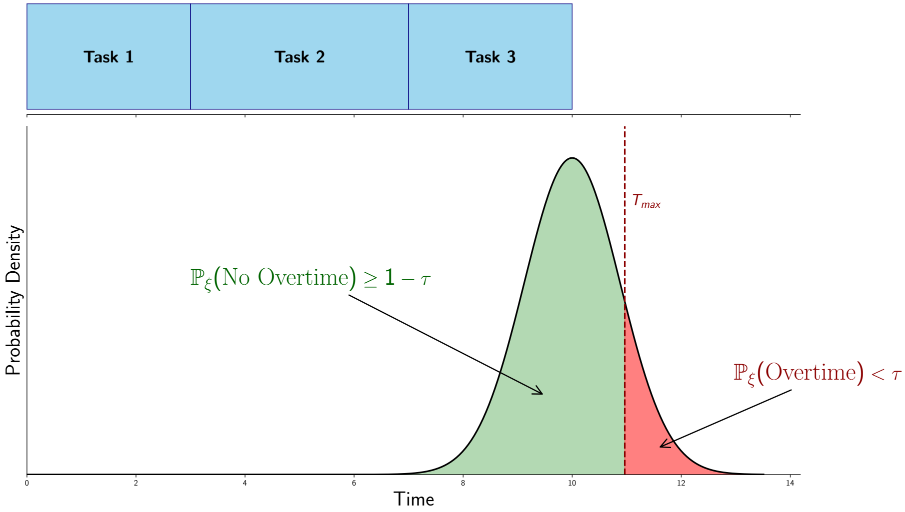
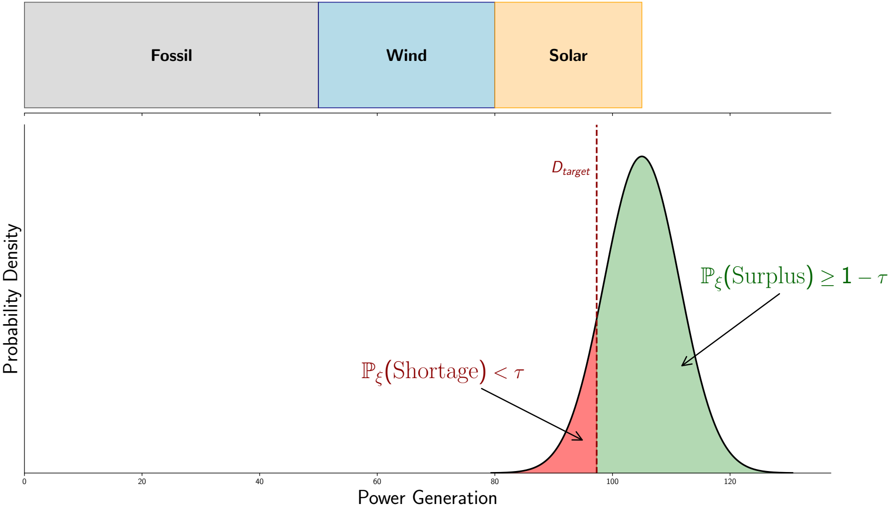
Applications: Scheduling with Uncertain Durations



Applications: Energy Demand Satisfaction



Overview

Setting and Challenges

CCSP with Finite Support

Assume

$$\xi \in \{\xi^s \in \mathbb{R}^d : s \in S\}$$

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Parameters & Variables

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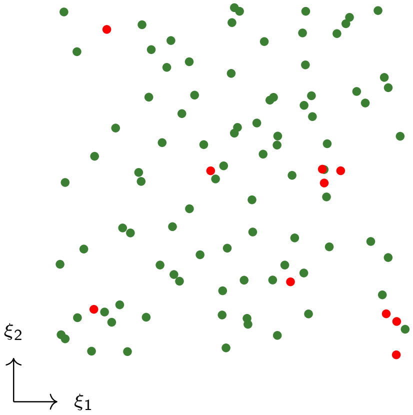
Feasibility with Finite Support



For $\underline{x}$, assume $\exists\, {\color{green}\blacksquare}, {\color{red}\blacksquare} \subseteq \mathcal{E}$,

$$\underline{x} \in X(\xi), \;\; \forall \xi \in {\color{green}\blacksquare}$$

and

$$\underline{x} \notin X(\xi), \;\; \forall \xi \in {\color{red}\blacksquare}$$

Then, $\underline{x}$ is feasible if

$$\mathbb{P}(\xi \in {\color{green}\blacksquare}) \geq 1 - {\color{red}\tau}$$

or equivalently

$$\mathbb{P}(\xi \in {\color{red}\blacksquare}) \leq {\color{red}\tau}$$

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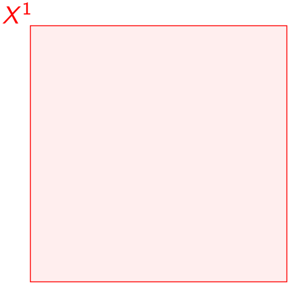

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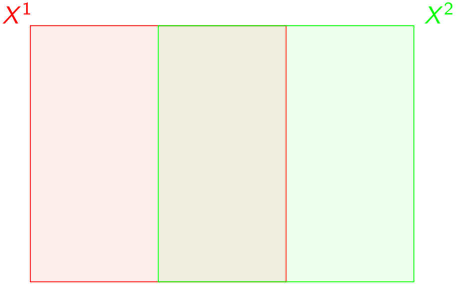

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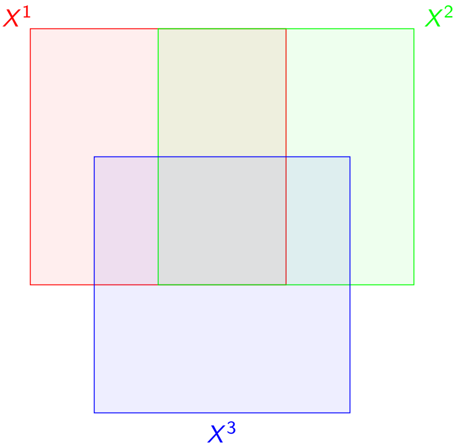

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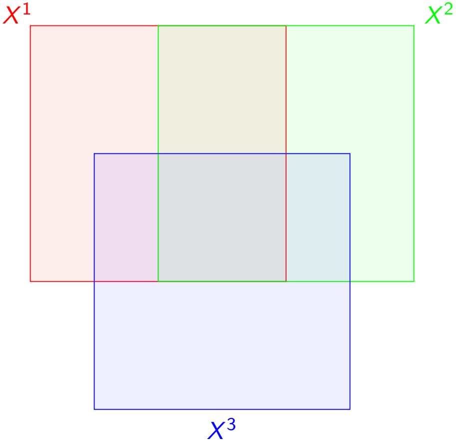

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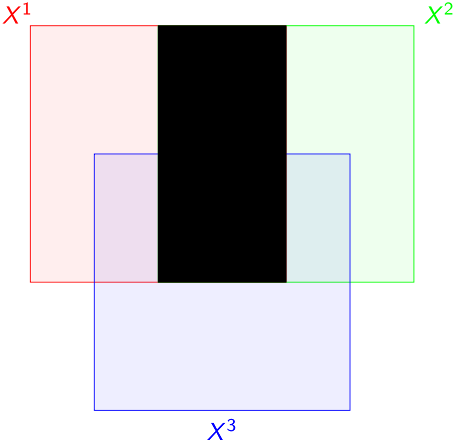

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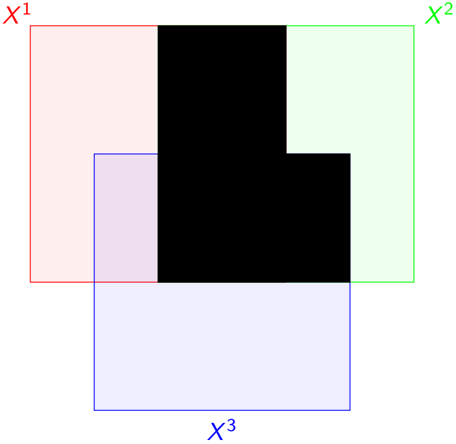

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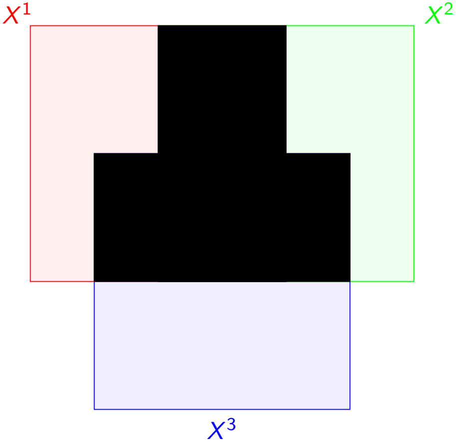

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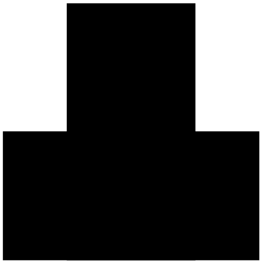

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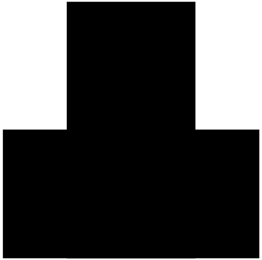

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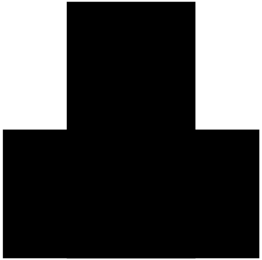

$\rightarrow$ Noncovex nonsmooth x-space and $\binom{|S|}{\lfloor \tau |S| \rfloor}$ operations

$\rightarrow$ NP-hard to solve even for simplified cases (Luedtke et al. 2010)

Linear CCSP with Equiprobable Scenarios

Assumptions

$$p_s = \frac{1}{|S|} \qquad X^s = \{x \in \mathbb{R}^n : \; A^s x \geq b^s\} \qquad f(x) = c^\top x$$

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Big-M Coefficients: Reformulation

Reformulation

$$z_s = \mathbb{1}\left(A^s x \geq b^s\right) \qquad \rightarrow \qquad A^s x - b^s \geq \mathrm{M}(1 - z_s)$$

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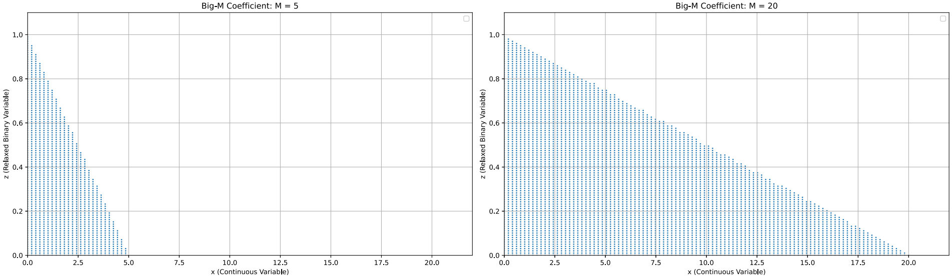
Challenge 2: Relaxation Strength

Example

$$z = \mathbb{1}(x \leq 0) \quad \longrightarrow \quad x \leq M(1 - z)$$

Illustration



State of the Art

Overview: Strengthening Techniques

Cutting planes

Mixing-Set (Luedtke 2014; Song et al. 2014), Quantile (Xie et al. 2018), Infeasible Irreducible Subsystems (Tanner et al. 2010), Multi-Disjunctive (Cattaruzza et al. 2024)

Big-M tightening

Instance-Specific (Song et al. 2014), Generic (Qiu et al. 2014)

Convex Approximations

Outer (Ahmed et al. 2017), Inner (Jiang et al. 2022; Nemirovski et al. 2007)

Alternative Approaches

Adaptive Partitioning (Roland et al. 2023)

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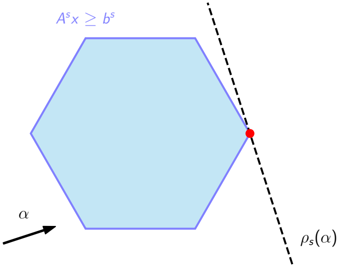
Single Scenario Cost

$$\rho_s(\alpha) := \min \quad \alpha^\top x$$
$$\text{s.t.} \quad A^s x \geq b^s$$



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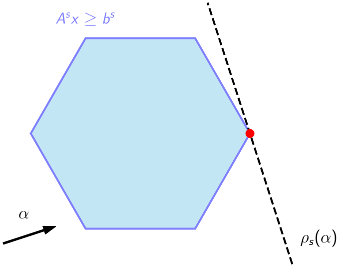

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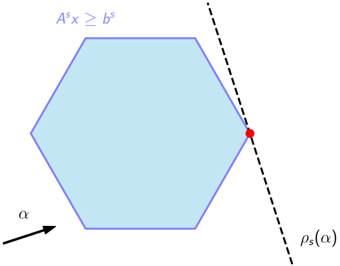

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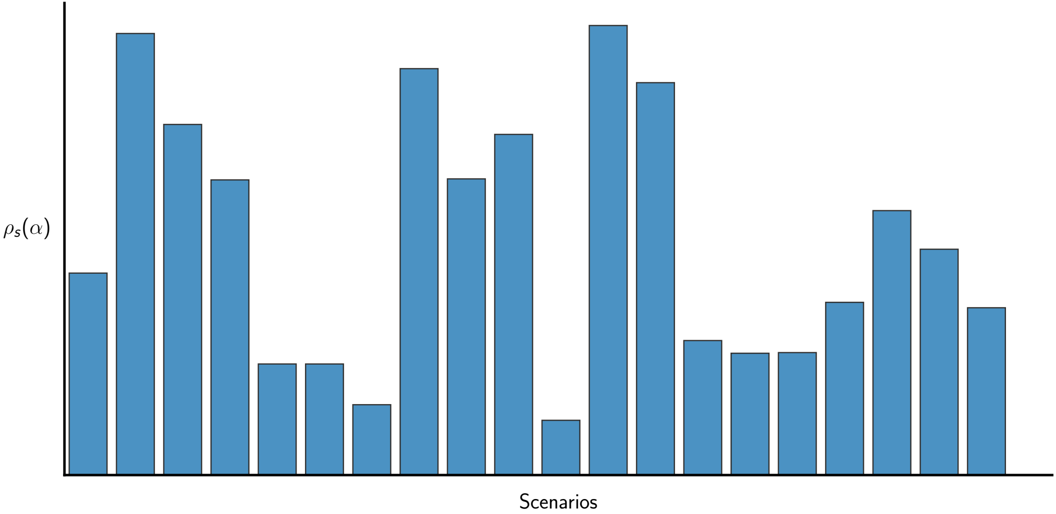
Generalization

$$\alpha^\top x \geq \max_{s \in S} \{\rho_s(\alpha) : z_s = 1\}$$

The Quantile Bound II



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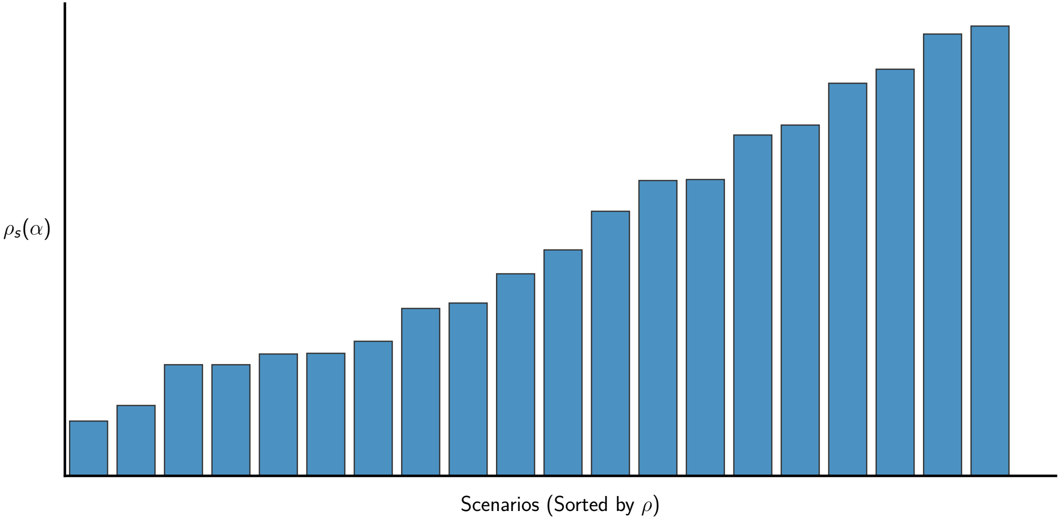

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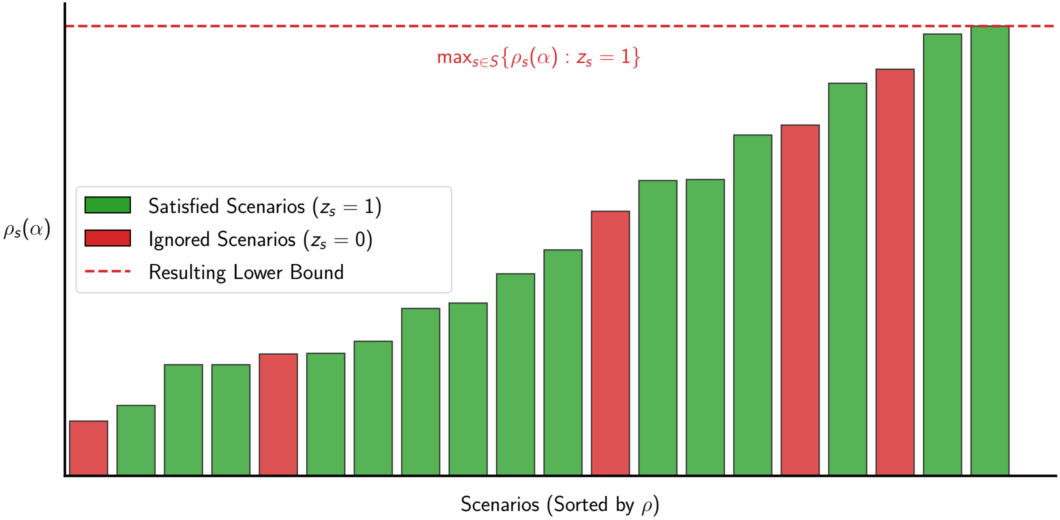

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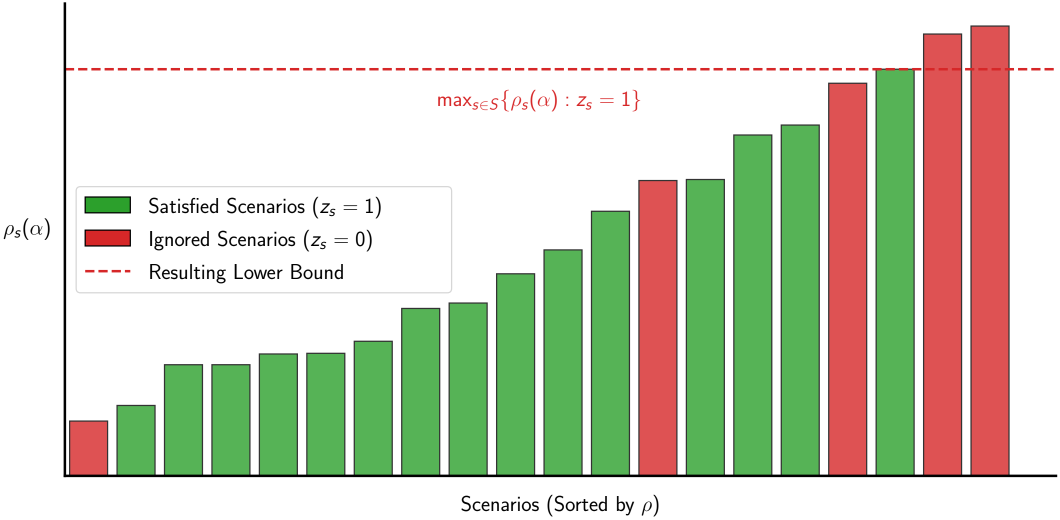

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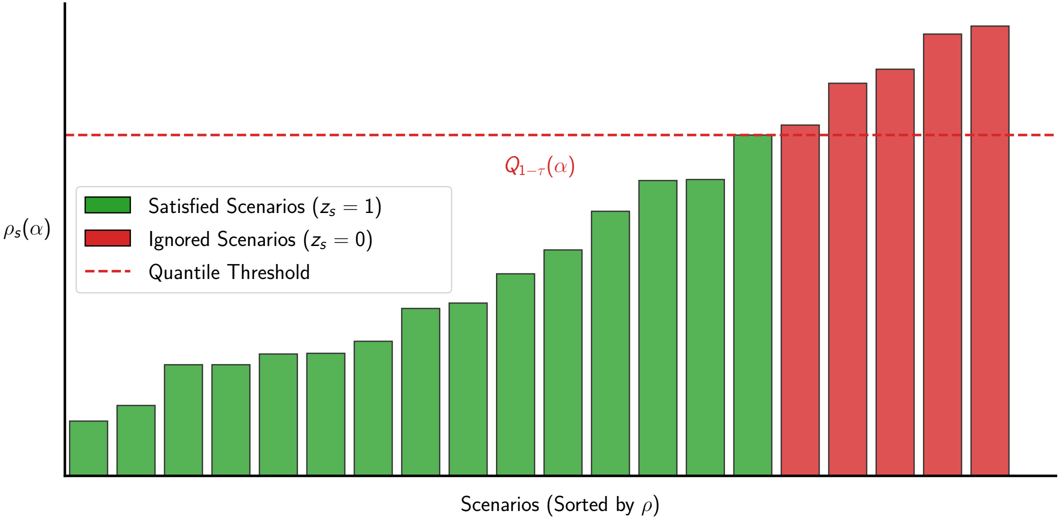

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4. **Valid Inequality**

$$\alpha^\top x \geq \max_{s \in S}\{\rho_s(\alpha) : z_s = 1\}$$

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2. **Compute** $\rho_s(\alpha)$ **for all** $s \in S$
3. **Sort** $\phi : S \to S$ **such that**

$$\rho_{\phi_1}(\alpha) \leq \rho_{\phi_2}(\alpha) \leq \cdots \leq \rho_{\phi_{|S|}}(\alpha)$$

4. **Valid Inequality**

$$\alpha^\top x \geq \max_{s \in S} \{\rho_s(\alpha) : z_s = 1\} \geq Q(\alpha) := \rho_{\phi_{|S| - \lfloor \tau |S| \rfloor}}(\alpha)$$

Quantile Inequalities: Exact Separation

- How to choose α?

$\rightarrow$ Separation (finding the best α) in NP (Xie et al. 2018)

Quantile Inequalities: Heuristic Separation

- Computing Q $\rightarrow$ Select α + $|S|$ LPs + `argsort`

$\rightarrow$ Heuristics are expensive ...

$\rightarrow$ Litterature: bounding $\rho_s(\alpha)$ (Song et al. 2014)

Strengthened Mixing Set

Observation: We can model the on-off relationship between x and z

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$$M(\alpha) = \left\{ (x, z) : \ \alpha^\top x \geq Q(\alpha) - (Q(\alpha) - \rho_s(\alpha)) z_s, \ \forall s \text{ above } Q(\alpha) \right\}$$

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Cases

→ If $z_s = 0$:

$$\alpha^\top x \geq Q(\alpha)$$

→ If $z_s = 1$:

$$\alpha^\top x \geq \rho_s(\alpha)$$

Mixing Sets: Closure

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For $\alpha \in \mathbb{R}^n$ (Qiu et al. 2014)

$$\mathrm{Proj}_x(\mathrm{conv}(M(\alpha))) = \{x \in \mathbb{R}^n_+ : \ \alpha^\top x \geq Q(\alpha)\}$$

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Basic Mixing Closure

$$\mathcal{BM} := \bigcap_{\alpha \in \mathbb{R}^n} \mathrm{Proj}_x(\mathrm{conv}(M(\alpha))) = \bigcap_{\alpha \in \mathbb{R}^n} \{x \in \mathbb{R}^n_+ : \ \alpha^\top x \geq Q(\alpha)\} =: \mathcal{Q}$$

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$\rightarrow$ **What about the Real Mixing Closure?**

$$\mathcal{M} := \mathrm{Proj}_x\left(\bigcap_{\alpha \in \mathbb{R}^n} \mathrm{conv}(M(\alpha))\right) \qquad \mathcal{M} \subseteq \mathcal{BM} = \mathcal{Q}$$

Mixing Sets: Exact Separation

- How to choose α?

- Recall: $\mathcal{BM} = \mathcal{Q}$

$\rightarrow$ As hard to separate as Q-inequalities

Mixing Sets: Heuristic Separation

- Computing $\rho_s(\alpha), Q(\alpha) \quad \rightarrow \quad$ Select α + $|S|$ LPs + `argsort`

$\rightarrow$ Heuristics are expensive ...

$\rightarrow$ Litterature: bounding $\rho_s(\alpha)$ (Song et al. 2014)

Observation

- We consider the $Q(\alpha)$ scenario, but all other scenarios are ignored

- We haven't used the constraints $A^s x \geq b^s$

Multi Dimensional Disjunctive Inequalities

Disjunctive Aspect of CCSPs

Simple Example

$$S = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\} \qquad \tau = 0.4 \rightarrow \lfloor \tau |S| \rfloor = 2 \qquad |\bar{S}| = 4$$

Property

$\rightarrow$ At least $|\bar{S}| - \lfloor \tau |S| \rfloor$ scenarios inside $\bar{S}$ are satisfied.

Remainder

$$|\hat{S}(\bar{x})| \geq |\bar{S}| - \lfloor \tau |S| \rfloor$$

Temporary Assumption

$$A^s x \geq b^s \qquad \rightarrow \qquad a^s x \geq b^s$$

Multi Dimensional Inequalities: Local Validity

Arbitrary subset

$$\bar{S} \subseteq S \quad \text{where} \quad |\bar{S}| \geq \lfloor \tau |S| \rfloor + 1$$

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Existence

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Summation

$$\sum_{i \in [n]} \left(\sum_{s \in \hat{S}(\bar{x})} (a_i^s) \right) \bar{x}_i \geq \sum_{s \in \hat{S}(\bar{x})} (b^s)$$

Multi Dimensional Inequalities: Upper Bound II

Upper Bound

$$\sum_{s\in\hat{S}(\bar{x})} (a_i^s) \le U_i(\bar{S}) = \max_{w} \quad \sum_{s\in\bar{S}} a_i^s w^s$$

$$\text{s.t.} \quad \sum_{s\in\bar{S}} w^s = |\bar{S}| - \lfloor \tau |S| \rfloor$$

$$w^s \in [0,1], \quad s \in \bar{S}$$

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$$\text{s.t.} \quad \sum_{s\in\bar{S}} w^s = |\bar{S}| - \lfloor \tau |S| \rfloor$$
$$w^s \in [0,1], \quad s \in \bar{S}$$

Lower Bound

$$\sum_{s\in\hat{S}(\bar{x})} (b^s) \geq L_i(\bar{S}) = \min_{w} \sum_{s\in\bar{S}} b^s w^s$$
$$\text{s.t.} \quad \sum_{s\in\bar{S}} w^s = |\bar{S}| - \lfloor \tau |S| \rfloor$$
$$w^s \in [0,1], \quad s \in \bar{S}$$

Multi Dimensional Inequalities: Generalization I

Arbitrary subset

$$\bar{S} \subseteq S \quad \text{where} \quad |\bar{S}| \geq \lfloor \tau |S| \rfloor + 1$$

Valid Inequality

$$\sum_{i \in [n]} U_i(\bar{S}) x_i \geq L(\bar{S})$$

Parameters

$$\begin{aligned} U_i(\bar{S}) = \max_w \quad & \sum_{s \in \bar{S}} a_i^s w^s \\ \text{s.t.} \quad & \sum_{s \in \bar{S}} w^s = |\bar{S}| - \lfloor \tau |S| \rfloor \\ & w^s \in [0,1], \quad s \in \bar{S} \end{aligned}$$

$$\begin{aligned} L(\bar{S}) = \min_w \quad & \sum_{s \in \bar{S}} b^s w^s \\ \text{s.t.} \quad & \sum_{s \in \bar{S}} w^s = |\bar{S}| - \lfloor \tau |S| \rfloor \\ & w^s \in [0,1], \quad s \in \bar{S} \end{aligned}$$

Multi Dimensional Inequalities: Generalization II

Remove Assumption

$$a^s x \geq b^s \qquad \rightarrow \qquad (\lambda^s)^\top A^s x \geq (\lambda^s)^\top b^s$$

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Theorem (Conforti et al. 2014)

Inequality $\mu x \geq \delta$ is valid $\quad \Leftrightarrow \quad$ *there exits $\lambda^s \in \mathbb{R}^n$ such that*

$$(\lambda^s)^\top A = \mu \quad \textit{and} \quad (\lambda^s)^\top b \geq \delta.$$

Multi Dimensional Inequalities: Summary

1. **Multiplier λ^s**

   $\rightarrow$ One valid inequality ($\lambda^s A^s x \geq \lambda^s b^s$) for every $s \in \bar{S}$

2. **Subset $\bar{S}$**

   $\rightarrow$ Globally valid upper and lower bound (U_i and L)

Multi Dimensional Inequalities: Closure

Multi Dimensional Closure

$$\mathcal{MD} := \bigcap_{\bar{S},\lambda} \left\{ x \in \mathbb{R}^n_+ : \sum_{i\in[n]} U_i(\lambda^s, \bar{S}) x_i \geq L(\lambda^s, \bar{S}) \right\}$$

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Proper Subset

$$\mathcal{MD} \subset \mathcal{Q} = \mathcal{BM}$$

→ $(\subseteq)$ $\mathcal{Q}$ special case where $|\bar{S}| = \lfloor \tau |S| \rfloor + 1$

→ $(\subset)$ Can construct instances where inclusion is strict.

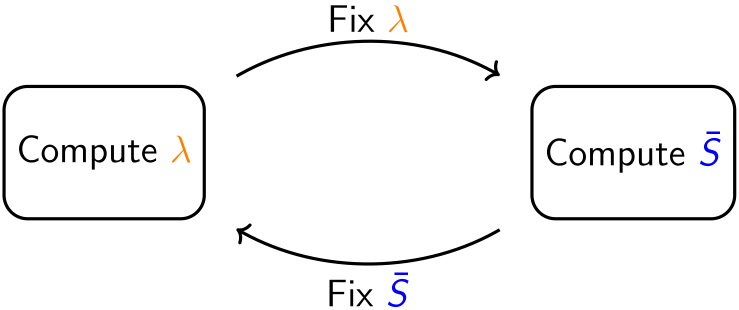
Multi Dimensional Inequalities: Exact Separation

- Separation in $\mathtt{NP}$

$\rightarrow$ Computing optimal λ^s in $\mathtt{P}$ (LP)

$\rightarrow$ Computing good **heuristic** $\bar{S}$ in $\mathtt{P}$ (Greedy Algorithm)

Separation Heuristic: ADM-Like



$\rightarrow$ We carry out one iteration per direction

Separation Heuristics: Constraint-wise

1. **One constraint j at a time**

$$A^s x \geq b^s \qquad \rightarrow \qquad A^s_{j\cdot} x \geq b^s_j$$

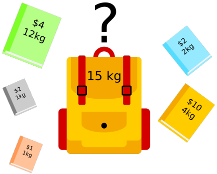
2. **Compute $\bar{S}$ for every j**

3. **Select most violated j**

Numerical Experiments

Multidimensional Knapsack Instances: Generation

Base Instance



$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Perturbation

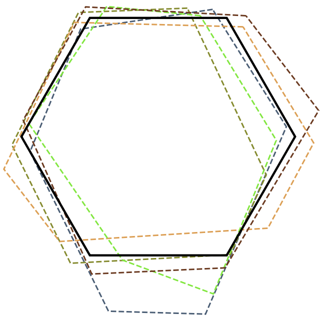
$$a_{ij}^s \sim \mathcal{N}(\mu = a_{ij}, \sigma^2 = 0.1 \cdot a_{ij})$$

Big-M (Naive)

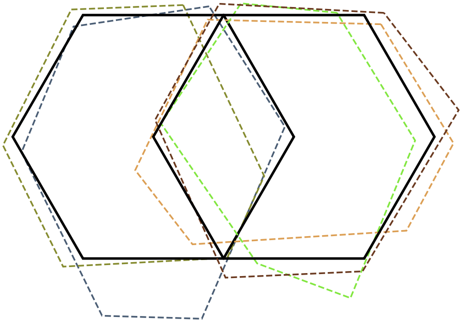
$$M_{j.}^s = A_{j.}^s \cdot 1$$

Multidimensional Knapsack Instances: Visualization

Litterature (`Lit`)



New (`New`)



Cutting Plane Method: Setup

- **Instances**: Perturbed Multidimensional Knapsack
- **Software**: Python, Gurobi, Cython
- **Benchmark**: Quantile and Mixing based on Song et al. 2014

$\rightarrow$ Choose α based on violated constraints

$\rightarrow$ Knapsack specific: tight lower bound on $\rho_s(\alpha)$

Cutting Plane Method: Results

Instances			MD-VIs							Mixing-set			Quantile		
			ADM				C-Wise								
(n, m)	Gen.	$\lvert S\rvert$	s(λ)	T	it.	v (%)	T	it.	v (%)	T	it.	v(%)	T	it.	v (%)
mk-20-10	lit	100	0.87	0.18	18.80	**30.21**	0.46	4.00	28.05	0.11	4.80	25.07	0.11	4.20	21.18
		500	0.73	0.42	13.80	**25.50**	0.59	4.00	23.47	0.52	6.20	22.10	0.51	3.80	19.84
		1000	0.89	0.81	15.00	**27.00**	0.65	3.80	25.14	1.09	9.00	24.47	1.04	5.00	21.89
		3000	0.88	3.27	11.80	**25.80**	0.76	3.40	24.77	3.88	17.80	22.99	3.23	4.80	20.82
	new	100	0.33	0.23	21.60	**24.03**	0.63	3.60	4.30	0.12	5.20	16.30	0.12	2.40	0.92
		500	0.33	0.43	30.80	**27.27**	0.80	4.20	10.97	0.60	8.00	18.60	0.58	4.00	1.55
		1000	0.29	0.79	15.80	**25.10**	0.86	3.00	11.31	1.17	8.40	15.92	1.09	4.20	2.65
		3000	0.31	3.04	22.20	**25.20**	0.74	3.80	5.21	3.72	18.20	16.67	3.40	3.80	2.11
mk-40-30	lit	100	0.77	0.62	39.80	**27.65**	0.73	4.20	25.05	1.36	6.40	22.78	1.37	5.60	15.81
		500	0.79	1.27	21.00	**26.13**	0.61	3.80	23.66	6.72	7.40	19.72	6.67	4.60	16.04
		1000	0.73	3.17	47.40	**28.19**	0.76	4.20	25.46	13.55	12.40	22.74	11.83	6.20	18.94
		3000	0.79	8.96	20.80	**25.59**	0.78	3.60	23.25	41.06	17.20	19.05	38.16	5.20	15.85
	new	100	0.23	0.76	63.80	**26.08**	0.94	2.60	4.18	1.25	6.80	17.78	1.37	3.00	0.99
		500	0.26	1.15	31.80	**23.30**	0.89	3.20	6.42	6.89	6.80	14.27	6.74	4.00	1.67
		1000	0.22	2.64	39.80	**24.47**	0.89	3.00	4.40	13.18	10.80	15.73	12.66	4.20	2.14
		3000	0.25	8.75	32.20	**25.08**	0.90	3.80	7.13	40.25	19.80	14.93	36.87	4.00	1.77

Cutting Plane Method + MIP: Setup

- **Instances**: Perturbed Multidimensional Knapsack
- **x Variables**: Binary
- **Software**: Python, Gurobi, Cython
- **Time**: 2 hours

Cutting Plane Method + MIP: Results

Instances				MD-VIs			
(n, m)	Gen.	$\vert S\vert$	MIP	`ADM`	`C-Wise`	Mixing-set	Quantile
mk-20-10	`lit`	100	9.28s	7.47s	**4.31s**	6.34s	5.49s
		500	110.42s	46.33s	**21.53s**	60.40s	39.13s
		1000	642.60s	**111.55s**	120.40s	260.68s	164.71s
		3000	12.03%(1)	2.19%(4)	**696.69s**	4.60%(4)	1500.19s
	`new`	100	14.43s	13.01s	8.40s	**8.01s**	13.11s
		500	91.75s	**62.04s**	70.99s	72.04s	91.51s
		1000	388.66s	**254.66s**	296.24s	356.02s	442.39s
		3000	41.46%(0)	**32.92%(0)**	33.83%(1)	37.82%(0)	52.73%(0)
mk-40-30	`lit`	100	10.17%(0)	9.16%(1)	**8.32%(1)**	11.73%(0)	12.05%(0)
		500	29.22%(0)	5.73%(2)	**1.81%(4)**	14.44%(0)	15.76%(0)
		1000	52.17%(0)	**20.85%(0)**	24.03%(0)	29.35%(0)	34.95%(0)
		3000	41.63%(0)	**15.07%(0)**	18.14%(0)	24.66%(0)	29.13%(0)
	`new`	100	19.70%(0)	22.23%(0)	**18.79%(0)**	21.98%(0)	19.05%(0)
		500	39.24%(0)	**20.45%(0)**	32.42%(0)	27.21%(0)	34.72%(0)
		1000	48.89%(0)	**29.81%(0)**	55.00%(0)	39.56%(0)	52.94%(0)
		3000	49.05%(0)	**27.68%(0)**	48.18%(0)	42.26%(0)	50.00%(0)

Conclusion

Problem: Linear Chance Constrained Stochastic Programs with Finite Support

Challenges: Non-convexity, Weak LP relaxations

State of the Art: Quantile and Mixing Set Inequalities

Concept: Disjunctive Aspect of CCSPs

Theoretical Dominance:

$$\mathcal{MD} \subset \mathcal{Q} = \mathcal{BM}$$

Computational Dominance: Better performance + More robust

Some Open Questions

- Heuristics for choosing α for quantile and mixing set inequalities?
- What about the Real Mixing Closure $\mathcal{M}$?
- Where does $\mathcal{M}$ stand in $\mathcal{MD} \subset \mathcal{Q} = \mathcal{BM}$?
- How close is $\mathcal{MD}$ to conv(F)?
- Do stronger valid inequalities exist?

Thank You!